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THE RESPONSE OF A FORCED OSCILLATOR UNDER THE EFFECT OF A PAIR OF SET-UP ELASTIC STOPS

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1. INTRODUCTION

Various methods have been used for the study of non-linear dynamical systems. The same methods can be used for piecewise-linear oscillators. By the simultaneous restitution model, which neglects the impact time so that the restoring force goes to infinity on impact, Shaw [1] studied a forced oscillator impacting on two rigid walls, while Cone and Zadoks [2] considered the vibro-impact system with friction damping.

In order to include a finite impact time, some authors have assumed a model with elastic stops having a continuous- and piecewise-linear restoring force. For example, Hu [3, 4] analyzed the grazing-induced bifurcations of harmonically forced trilinear oscillators and Natsiavas [5] developed a numerical scheme to locate the periodic motions and to determine their stability.

Another possible model is a harmonically forced oscillator with a pair of symmetrical set-up elastic stops. In this case the pre-loaded elastic stops in the oscillator results in a more complicated restoring force. This model is widely used in vibration control and vibration machinery. The elastic restoring force in this case is piecewise linear, but if the mass touches the set-up elastic stops, the restoring force undergoes a finite jump equal to the pre-load. Den Hartog [6] studied a harmonically forced, undamped oscillator with a pair of set-up springs, while Mahfouz and Badrakhan [7] studied numerically the chaotic motion of this type of oscillators. Hu [8] used the average approach to determine the qualitative changes of the system dynamics caused by the pre-load on the fundamental resonance of a harmonically forced oscillator, but with no comparison with numerical results. The same model is considered in this paper. The relative differential equation is

$$m\ddot{X}' + c'\dot{X}' + kX' + k\alpha g'(X') = 2f' \cos \Omega' t', \tag{1a}$$

where m is the mass quantity, $\Omega' \approx \sqrt{k/m}$ the excitation frequency, c' the linear damping coefficient, k the linear stiffness coefficient, f' the excitation semi-amplitude and α the ratio of the stiffness of an elastic stop to the linear stiffness. Moreover, $k\alpha g'(X')$ represents the restoring force of the symmetrically set-up elastic stops with set-up amount a' and clearance b:

$$g'(X') = \begin{cases} 0, & |X'| \leqslant b, \\ X' + (a' - b)\operatorname{sgn} X', & |X'| > b. \end{cases} \tag{1b}$$

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The asymptotic perturbation (AP) method [9–11] is used in order to determine an approximate analytic solution of the system (1a, b). The AP method is based on the introduction of a slow time and balancing of harmonic terms with a simple iteration. In a certain sense, the AP method can be considered as an attempt to link the most useful characteristics of harmonic balance and multiple-scale methods [12, 13]. The formal perturbation solution is carried out to the lowest order approximation.

The main motivation of this research is to show the feasibility of the AP method in studying a dynamic system with a discontinuous vector field. Moreover, the traditional harmonic balancing method is only sufficient to construct approximate steady state solutions for non-linear oscillators, but it cannot furnish any information about stability, whilst this information is easily obtained by the AP method.

The paper is arranged as follows. In Section 2 a review of the AP method is presented and it is demonstrated that in a first approximation the behaviour of the solution can be described by means of a model system of differential equations, which describe the slow flow of amplitude and phase of the approximate solution. In Section 3, frequency- and excitation amplitude-response curves are derived and the analytic results compared with the numerical integration. The response of the oscillator appears more complex compared with the simple linear oscillator. Jump and hysteresis phenomena are present in many regions of parameter space. The most qualitative change due to the stops is that the periodic motion of the oscillator loses its stability when it begins to touch the set-up elastic stops with a decrease in the excitation frequency. Finally, in the last section, the most important results and some possible generalizations of the present study are summarized.

2. THE APPROXIMATE SOLUTION OF THE AP METHOD

Via the rescalings

$$t = \sqrt{\frac{m}{k}}\,t', \qquad X = \frac{X'}{b}, \qquad c = \frac{c'}{\sqrt{mk}}, \qquad a = \frac{a'}{b}, \qquad f = \frac{f'}{kb}, \qquad \Omega = \Omega' \sqrt{\frac{m}{k}}. \tag{2}$$

Equations (1a, b) can be transformed into the dimensionless differential equation

$$\ddot{X} + X + c\dot{X} + \alpha g(X) = 2f \cos \Omega t, \tag{3a}$$

where

$$g(X) = \begin{cases} 0, & |X| \leqslant 1, \\ X + (a-1)\,\mathrm{sgn}\,X, & |X| > 1. \end{cases} \tag{3b}$$

The AP method used to calculate the approximate solution was first developed in References [9–11] and then in this section the main steps of this perturbation technique are described. First of all, the slow time

$$\tau = \varepsilon t, \tag{4}$$

is introduced in order to identify the temporal scale where the non-linear effects become consistent and not negligible. If $t \to \infty$, then $\varepsilon \to 0$, when τ assumes a finite value.

The parameter ε can be chosen as one of the small parameters of the system. It is assumed that $c = \varepsilon$ and the coefficients f, α are assumed to be of the order of ε (small damping, soft excitation and weak non-linearity). The detuned frequency σ is introduced in such a way that

$$\Omega = 1 + \varepsilon\sigma, \qquad \Omega^2 - 2\varepsilon\sigma - \varepsilon^2\sigma^2 = 1. \tag{5}$$

Equation (3a) yields

$$\ddot{X} + \Omega^2 X - 2\varepsilon\sigma X - \varepsilon^2\sigma^2 X + \varepsilon c\dot{X} + \varepsilon\alpha g(X) = 2\varepsilon f \cos \Omega t. \tag{6}$$

If $\varepsilon = 0$ in equation (6), it can be seen that it admits simple harmonic solutions $X(t) = A\exp(-\mathrm{i}\Omega t) + \text{c.c.}$, where A is a constant depending on the initial conditions and c.c. stands for complex conjugate. Non-linear effects induce a modulation of the amplitude A and the appearance of higher harmonics. The modulation is best described in terms of the rescaled variable τ, which accounts for the need to consider larger time scales, to obtain a non-negligible contribution from the non-linear term.

A solution $X(t)$ of equation (6) is sought which can be expressed by means of a power series in the expansion parameter ε:

$$X(t) = \sum_{n=-\infty}^{+\infty} \varepsilon^{\gamma_n}\psi_n(\tau, \varepsilon)\exp(-\mathrm{i}n\Omega t) \tag{7}$$

with $\gamma_n = |n| - 1$ for $n \neq 0$, and $\gamma_0 = 1$. Note that

$$\psi_n(\tau, \varepsilon) = \psi^*_{-n}(\tau, \varepsilon) \tag{8}$$

because $X(t)$ is a real function.

The assumed solution (7) can be considered to be a combination of the different harmonics, and solutions of the linear equation, i.e., of the equation obtained after neglecting all the non-linear terms, and the coefficients of this combination depend on τ and ε.

Equation (7) can be written more explicity as

$$X(t) = \varepsilon\psi_0(\tau;\varepsilon) + (\psi_1(\tau;\varepsilon)\exp(-\mathrm{i}\Omega t)$$
$$+ \varepsilon\psi_2(\tau;\varepsilon)\exp(-2\mathrm{i}\Omega t) + \text{c.c.}) + O(\varepsilon^2), \tag{9}$$

where c.c. stands for complex conjugate of the preceding terms. The functions $\psi_n(\tau, \varepsilon)$ depend on the parameter ε and it is supposed that the limit of the ψ_n's for $\varepsilon \to 0$ exists and is finite that they can be expanded in power series of ε, i.e.,

$$\psi_n(\tau;\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^{\mathrm{i}}\psi_n^{(i)}(\tau). \tag{10}$$

For simplicity in the following the abbreviations $\psi_n^{(0)} = \psi_n$ for $n \neq 1$ and $\psi_1^{(0)} = \psi$ for $n = 1$ are used.

Note that the variable change (4) implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\psi_n \exp(-\mathrm{i}n\Omega t)) = \left(-\mathrm{i}n\psi_n + \varepsilon\frac{\mathrm{d}\psi_n}{\mathrm{d}\tau}\right)\exp(-\mathrm{i}n\Omega t). \tag{11}$$

After inserting this expansion into equation (6), equations for every harmonic and for a fixed order of approximation are obtained. For $n = 1$ at the order of ε^0

$$(-\Omega^2 + \Omega^2)\psi = 0 \tag{12}$$

is obtained, which is identically satisfied, while at the order of ε and for $\rho > \frac{1}{2}$

$$-2\mathrm{i}\Omega\frac{\mathrm{d}\psi}{\mathrm{d}\tau} - 2\sigma\psi - \mathrm{i}c\Omega\psi + \frac{\alpha}{\pi}\left(\pi - 2\arcsin\left(\frac{1}{2\rho}\right) + \frac{(2a-1)}{2\rho^2}\sqrt{4\rho^2 - 1}\right) = f. \tag{13}$$

Substituting the polar form

$$\psi(\tau) = \rho(\tau)\exp(\mathrm{i}\vartheta(\tau)) \tag{14}$$

into equation (13), the separating real and imaginary parts, the following model system is obtained:

$$\frac{\mathrm{d}\rho}{\mathrm{d}\tau} = -\frac{c}{2}\rho + \frac{f}{2\Omega}\sin\vartheta, \tag{15a}$$

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}\tau} = \frac{\sigma}{\Omega} + \frac{f\cos\vartheta}{2\Omega\rho} - \frac{\alpha}{2\pi\Omega}\left(\pi - 2\arcsin\left(\frac{1}{2\rho}\right) + \frac{(2a-1)}{2\rho^2}\sqrt{4\rho^2 - 1}\right). \tag{15b}$$

In the case of $\rho \leqslant \frac{1}{2}$ (linear oscillator) it is sufficient to take $\alpha = 0$.

From equation (9) it can be seen that the approximate solution is

$$X(t) = 2\rho\cos(\Omega t - \vartheta) + O(\varepsilon). \tag{16}$$

The validity of the approximate solution should be expected to be restricted on bounded intervals of the τ-variable and on time scale $t = O(1/\varepsilon)$. If one wishes to construct solutions on intervals such that $\tau = O(1/\varepsilon)$ then the higher order terms must be included, because they will in general affect the solution.

In the next section the equilibrium points of equations (15a, b) will be considered which correspond to periodic orbits of the starting system (6) and subsequently the approximate solution (16) will be compared with numerical results.

3. EXCITATION AMPLITUDE AND FREQUENCY-RESPONSE CURVES

Equations (15a, b) are invariant under the transformation $f \to -f$, $\vartheta \to \vartheta - \pi$, and hence possess the corresponding symmetry. Thus if there is an equilibrium point at (f, ϑ_0), then there is also one at $(-f, \vartheta_0 - \pi)$. In order to simplify the following analysis, only half of the system is considered. If it is stated that the system contains an equilibrium point, then it actually contains two equilibria, the other one being located at the symmetrical position under the above-mentioned transformation. Firstly, note that the linear oscillator ($\alpha = 0$) possesses a stable equilibrium point given by

$$\rho_0 = \frac{f}{\sqrt{c^2\Omega^2 + 4\sigma^2}}, \qquad \vartheta_0 = \arctan\left(\frac{-c\Omega}{2\sigma}\right). \tag{17}$$

The small amplitude vibrations are always stable.

Next, the position and the stability of non-linear vibration with large amplitude only are considered, i.e., the case of $\rho > \frac{1}{2}$. The equilibrium points (ρ_0, ϑ_0) can be obtained from Equations (15a, b) and satisfy the equations

$$\Omega^2 c^2 \rho_0^2 + 4\rho_0^2\left(\sigma - \frac{\alpha}{2\pi}\left(\pi - 2\arcsin\left(\frac{1}{2\rho_0}\right) + \frac{(2a-1)}{2\rho_0^2}\sqrt{4\rho_0^2 - 1}\right)\right) = f^2, \tag{18a}$$

$$\vartheta_0 = \arctan\left(\frac{2\pi\Omega c\rho_0^2}{\alpha(2\pi\rho_0^2 - 4\rho_0^2\arcsin(1/2\rho_0) + (2a-1)\sqrt{4\rho_0^2 - 1}) - 4\pi\sigma\rho_0^2}\right) \tag{18b}$$

which must be solved numerically, for example by the Newton–Raphson method.

In order to establish the stability of steady state solutions, small perturbations are superimposed in the amplitudes and the phases on the steady state solutions and the resulting equations are then linearized. Subsequently, the eigenvalues of the corresponding system of first order differential equations with constant coefficients (the Jacobian matrix) are considered. A positive real root indicates an unstable solution, whereas if the real parts of the eigenvalues are all negative then the steady state solution is stable.

Results of stability analysis are given in Figure 1 in the parameter space $(a, f/\alpha)$ = (set-up, excitation amplitude-to-non-linearity ratio). Four regions (I, II, III and IV), exist in which different fundamental resonance periodic motions can exist. Figure 2 shows four frequency-response curves corresponding to typical parameter combinations in regions I–IV. The most important characteristic is that a stable periodic motion loses its stability if the oscillator touches the elastic stops and the frequency is decreasing. Another interesting feature is that the static stiffness of the system increases at first and then decreases with increase of the system displacement. As a result, in cases II and IV various stable periodic motions (multistability) coexist. The analytic approximate solution (16) have been

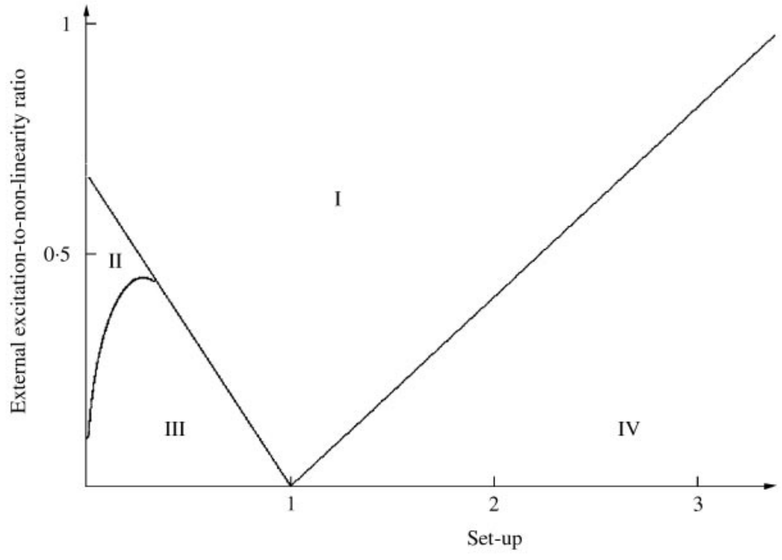

Figure 1. Four different types of resonance motions in the parameter space (set-up, excitation amplitude-to-non-linearity ratio) = $(a; f/\alpha)$.

compared with numerical results (represented by circles) from the direct integration of the original equations (3a, b). The fourth order Runge–Kutta method has been used with an integration step $h = \pi/20\Omega$. For the chosen values of parameters there is a satisfactory agreement between the two solutions.

For the parameter combinations in regions I, II and IV the frequency-response curve shows a more apparent effect of set-up elastic stops on the fundamental resonance, while in case III it is very similar to that of a system with elastic stops not preloaded [3–5]. Note that if a parameter is varied so as to pass from one region to another, the system loses its structural stability. Jump phenomena and hysteresis behaviour appear in regions I, III and IV. The analysis is in qualitative agreement with the results given by the average approach of Hu [8].

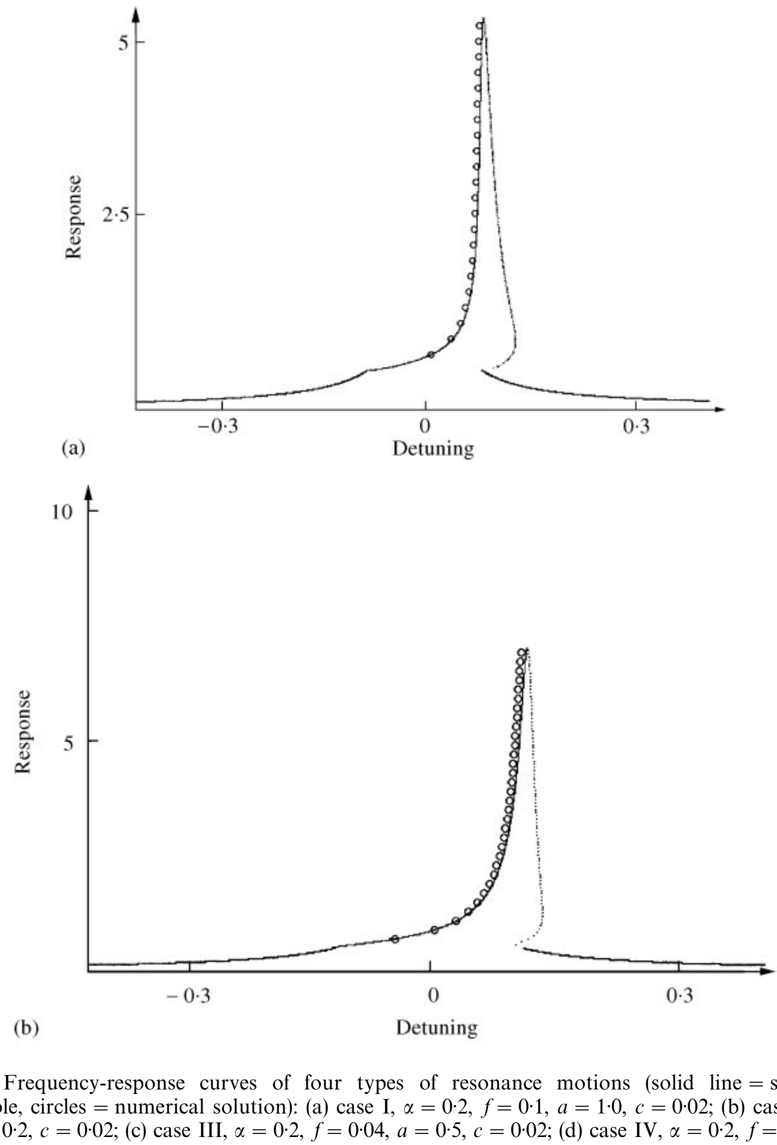

Figure 2. Frequency-response curves of four types of resonance motions (solid line = stable, dotted line = unstable, circles = numerical solution): (a) case I, $\alpha = 0{\cdot}2$, $f = 0{\cdot}1$, $a = 1{\cdot}0$, $c = 0{\cdot}02$; (b) case II, $\alpha = 0{\cdot}3$, $f = 0{\cdot}14$, $a = 0{\cdot}2$, $c = 0{\cdot}02$; (c) case III, $\alpha = 0{\cdot}2$, $f = 0{\cdot}04$, $a = 0{\cdot}5$, $c = 0{\cdot}02$; (d) case IV, $\alpha = 0{\cdot}2$, $f = 0{\cdot}04$, $a = 2{\cdot}0$, $c = 0{\cdot}02$.

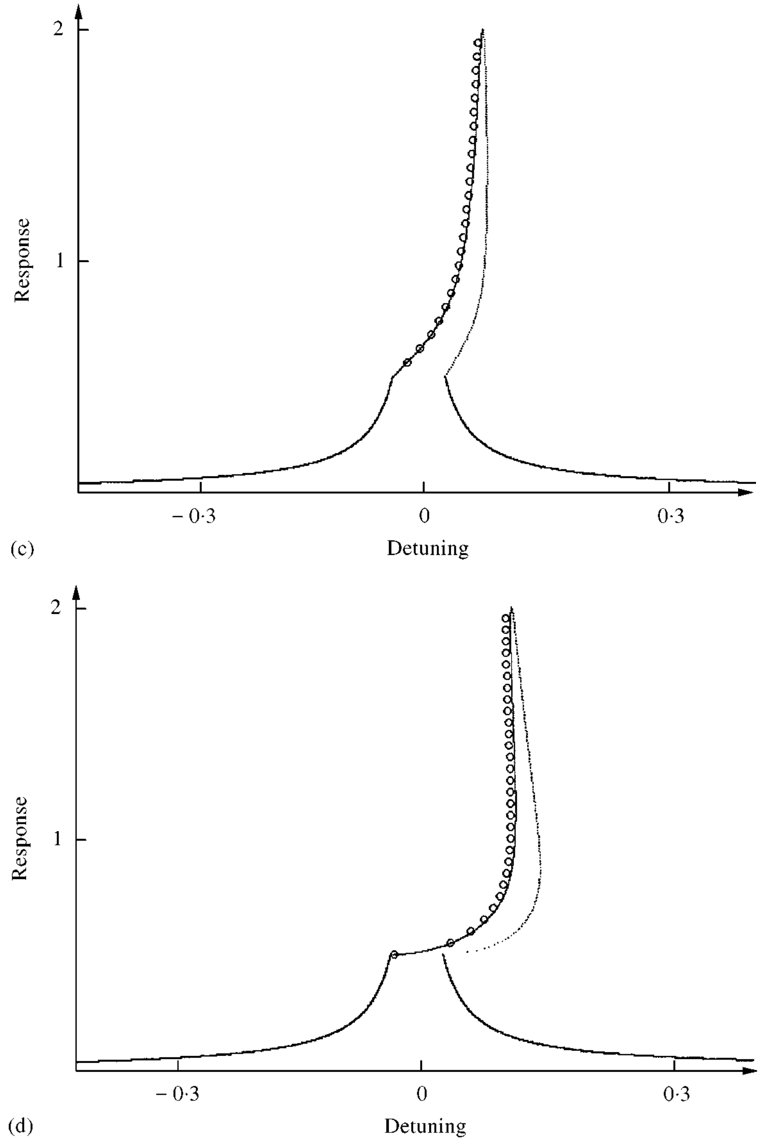

Figure 2. Continued.

Figure 3 shows the excitation amplitude-response curves. In Figure 3(a) the system passes through regions III, II and I, while in Figure 3(b) it passes from region IV to I. Also in these cases the typical hysteresis behaviour is observed.

4. CONCLUSION

The AP method has been used for the theory of approximate analytic solutions to non-linear oscillations of discontinuous piecewise-linear systems. This method is essentially based on two methods: the harmonic balance method and the perturbation (with two time

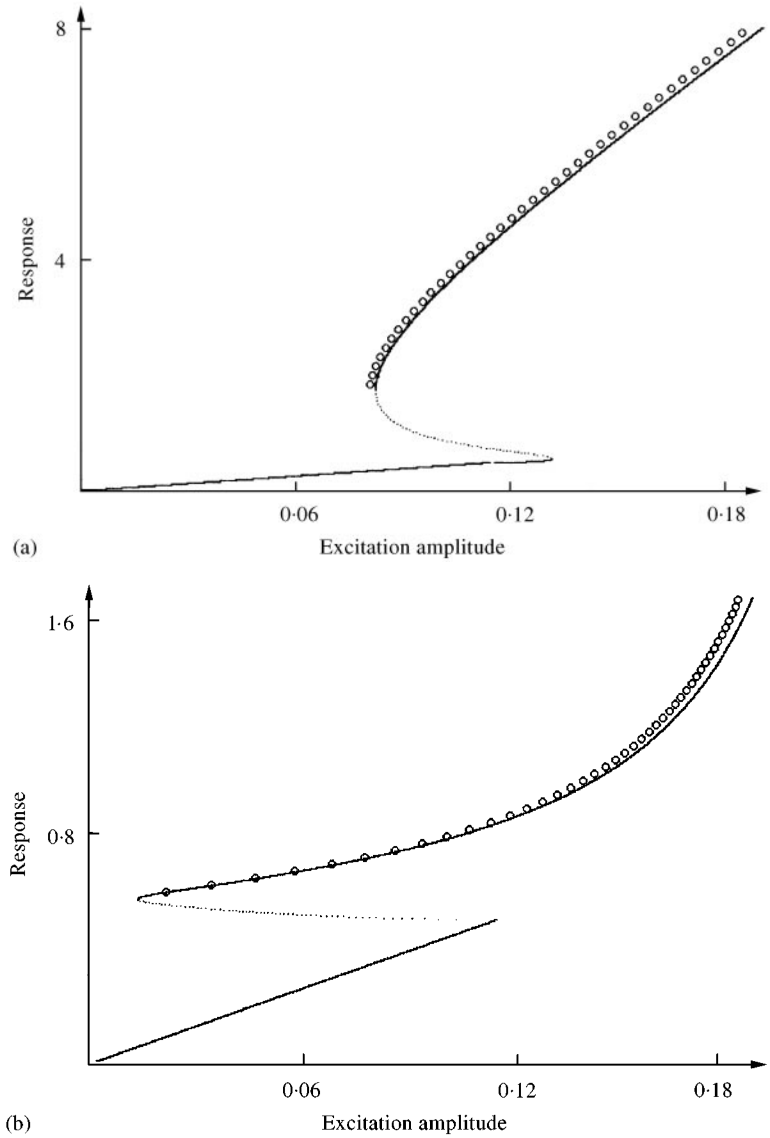

Figure 3. Excitation amplitude-response curves and hysteresis behaviour (solid line = stable, dotted line = unstable, circles = numerical solution): (a) $\alpha = 0{\cdot}2$, $\sigma = 0{\cdot}1$, $a = 0{\cdot}15$, $c = 0{\cdot}02$; (b) $\alpha = 0{\cdot}3$, $\sigma = 0{\cdot}1$, $a = 2{\cdot}0$, $c = 0{\cdot}02$.

scales) method. The AP method can describe precisely, for the parameter range considered, the behaviour of a weakly non-linear discontinuous piecewise-linear system for the case of fundamental resonance.

The AP method derives a system of non-linear model equations describing the modulation of the amplitude and of the phase of the oscillation. The dependence of the equilibrium points (periodic solutions of the original system) on the frequency and amplitude of the external force can be easily deduced.

Qualitative changes of the fundamental resonance appear as a consequence of the pre-load on the elastic stops in a haromonically forced oscillator. If the excitation frequency is decreasing, the periodic motion, when the elastic stops are touched, becomes unstable when the amplitude increases. Moreover, there are four types of persistent fundamental resonance and if the excitation amplitude is varied, it is possible to pass from one type to another.

Numerical results demonstrate the validity of the AP method and suggest that further study of other non-linear discontinuous piecewise-linear systems, under the effects of different resonances, should be carried out.

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